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Bright N-soliton solutions in terms of the triple Wronskian for the coupled nonlinear Schrödinger equations in optical fibers

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Abstract

The coupled nonlinear Schrödinger (NLS) equations are usually used to describe the dynamics of two-component solitons in optical fibers. Via the Darboux transformation, the coupled NLS equations of the Manakov type are found to have triple Wronskian solutions. Proof is finished by virtue of some new triple Wronskian identities. By solving the zero-potential Lax pair, the triple Wronskian solutions give the bright N-soliton solutions which are characterized by $3N$ complex parameters. To obtain an understanding of the asymptotic behavior of the bright N-soliton solutions with arbitrary N, some algebraic properties of the triple Wronskian are analyzed and an algebraic procedure is presented to derive the explicit expressions of the asymptotic solitons as $t \to \mp\infty$.

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1. Introduction

The coupled nonlinear Schrödinger (NLS) equations in the dimensionless form

$$\mathrm{i}u_t + \eta_1 u_{xx} + 2(\mu|u|^2 + \delta|v|^2)u = 0, \tag{1a}$$

$$\mathrm{i}v_t + \eta_2 v_{xx} + 2(\delta|u|^2 + \nu|v|^2)v = 0, \tag{1b}$$

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describe the simultaneous propagation of two nonlinear waves in a nonlinear, dispersive medium [1], where x and t are, respectively, the normalized distance and time coordinates, u and v are the slowly varying envelopes of two copropagating waves, η_j $(j = 1, 2)$ represents the group velocity coefficient in each component, μ and ν are the self-phase modulation coefficients and δ is the cross phase modulation coefficient. Recently, there have been efforts devoted to the two-component soliton dynamics described by equations (1*a*) and (1*b*) in the birefringent optical fiber, two-mode optical fiber and Kerr-like photorefractive medium [2]. Besides, the coupled NLS equations have arisen in other branches of physics such as plasmas [3], Bose–Einstein condensates [4] and hydrodynamics [5].

For an arbitrary choice of parameters, equations (1*a*) and (1*b*) are in general nonintegrable. However, equations (1*a*) and (1*b*) can pass the Painlevé test under two particular conditions: (1) $\eta_1 = \eta_2$, $\mu = \nu = \delta$; (2) $\eta_1 = -\eta_2$, $\mu = \nu = -\delta$ [6]. The first case, after suitable scale transformations, can be transformed into the celebrated Manakov model [7],

$$\mathrm{i}u_t + u_{xx} + 2(|u|^2 + |v|^2)u = 0, \tag{2a}$$

$$\mathrm{i}v_t + v_{xx} + 2(|u|^2 + |v|^2)v = 0, \tag{2b}$$

which has been derived as a key model for the lightwave propagation in the birefringent optical fibres [8]. It is mentioned that the Manakov model is the two-component vector generalization of the focusing NLS equation and can be exactly solved by the method of inverse scattering transform [7]. Therefore, from the analytical and integrable viewpoint, equations (2*a*) and (2*b*) are usually used as an approximate model of equations (1*a*) and (1*b*) for describing the two-component bright soliton propagation dynamics [2]. Also, equations (2*a*) and (2*b*) possess the following integrable properties: infinitely many local conservation laws [7, 9], an infinite-dimensional algebra of non-commutative symmetries [10], Lax pair-based Bäcklund transformation [11], Darboux transformation (DT) [12], bilinear representation [6, 13], analytical bright multi-soliton solutions [13, 14]. Although the integrability of equations (2*a*) and (2*b*) has been established from different points of view, the coupled NLS soliton collision dynamics seems nontrivial and requires to be further clarified due to the existence of the internal degrees of freedom of solitons [2]. The two-component structure in equations (2*a*) and (2*b*) can make the occurrence of some fascinating soliton collision behavior under certain parametric conditions, such as the intensity redistribution among colliding solitons in two components, amplitude-dependent phase shifts and changes in relative separation distances [14, 15]. It should be noted that the energy-exchange collisions of spatial and temporal vector solitons have been observed in the photorefractive media [16] and in the linearly birefringent optical fibers [17], respectively.

We note that by the Hirota bilinear method [18], DT [19] and Sato theory [20] the soliton solutions to a class of nonlinear evolution equations (NLEEs) can be represented in the form of a Wronskian. Via the Wronskian technique, one can not only write the soliton solutions in a simple, compact form, but also verify the solutions by direct substitution into the bilinear equations or by inductive use of the bilinear Bäcklund transformation [21, 22]. Moreover, the algebraic structure of the Wronskian provides a basis of studying the multi-soliton collision dynamics, as seen in [23], where the asymptotic behavior of a general class of line-soliton solutions to the Kadomtsev–Petviashvili II equation has been studied by expanding its τ-function as the Wronskian of N linearly independent combinations of M exponentials.

The soliton solutions in terms of the Wronskian have been revealed for a number of scalar NLEEs [18], but to our knowledge very little work has been done on the Wronskian-type representation of soliton solutions to the coupled NLEEs in the vector and matrix form [26].

The first purpose of this paper is to work out the Wronskian-type representation of the analytical bright N-soliton solutions to equations (2a) and (2b). With symbolic computation [24, 25], we will adopt the DT method to generate the triple Wronskian solutions to equations (2a) and (2b), and give a rigorous proof by developing some new determinantal identities (see lemma 3.3 as given below). By solving the zero-potential Lax pair, we will find that the bright N-soliton solutions to equations (2a) and (2b) can be expressed in terms of the triple Wronskian, which is, to our knowledge, reported here for the first time. In fact, the bright N-soliton solutions to equations (2a) and (2b) have already been obtained in [7, 13, 14], and the asymptotic analysis of the two- and three-soliton solutions has also been performed in [14]. However, until now, there has been an absence of the asymptotic analysis of the bright N-soliton solutions for arbitrary N. Thus, our second purpose is to present an algebraic procedure in explicitly obtaining the asymptotic expressions for the *generic* bright N-soliton solutions as $t \to \pm\infty$ on the basis of the triple Wronskian. This procedure will allow us to directly analyze the asymptotic behavior of arbitrary multi-soliton solutions to equations (2a) and (2b).

2. DT-based iterative algorithm

In the frame of the 3×3 Ablowitz–Kaup–Newell–Segur inverse scattering formulation [27], the Lax pair of equations (2a) and (2b) is of the form [7]

$$\Psi_x = L\Psi = \begin{pmatrix} -\mathrm{i}\lambda & u & v \\ -u^* & \mathrm{i}\lambda & 0 \\ -v^* & 0 & \mathrm{i}\lambda \end{pmatrix} \Psi, \tag{3a}$$

$$\Psi_t = M\Psi = \begin{pmatrix} -2\mathrm{i}\lambda^2 + \mathrm{i}(|u|^2 + |v|^2) & \mathrm{i}u_x + 2\lambda u & \mathrm{i}v_x + 2\lambda v \\ \mathrm{i}u_x^* - 2\lambda u^* & 2\mathrm{i}\lambda^2 - \mathrm{i}|u|^2 & -\mathrm{i}vu^* \\ \mathrm{i}v_x^* - 2\lambda v^* & -\mathrm{i}uv^* & 2\mathrm{i}\lambda^2 - \mathrm{i}|v|^2 \end{pmatrix} \Psi, \tag{3b}$$

where $\Psi = (\psi_1, \psi_2, \psi_3)^T$ (the superscript T signifies the vector transpose) is the vector eigenfunction, λ is the spectral parameter, the star denotes the complex conjugate and the compatibility condition $L_t - M_x + LM - ML = 0$ is exactly equivalent to equations (2a) and (2b).

The DT is such a kind of gauge transformation that leaves the form of a Lax pair invariant, and in general comprises the eigenfunction transformation and potential transformation [28]. According to the previous work in [12, 29], we know that equations (2a) and (2b) have the DT in terms of the first-order polynomial of λ, as follows.

Proposition 2.1. *Assume that $(f_1, g_1, h_1)^T$ is a solution of equations (3a) and (3b) with $\lambda = \lambda_1$. The DT $(\Psi, u, v) \to (\hat{\Psi}, \hat{u}, \hat{v})$ of the Lax pair (3) is given by*

$$\hat{\Psi} = (\lambda I_3 - S)\Psi, \qquad S = PJP^{-1},$$
$$J = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1^* & 0 \\ 0 & 0 & \lambda_1^* \end{pmatrix}, \qquad P = \begin{pmatrix} f_1 & -g_1^* & -h_1^* \\ g_1 & f_1^* & 0 \\ h_1 & 0 & f_1^* \end{pmatrix}, \tag{4}$$

$$\hat{u} = u + \frac{2\,\mathrm{i}(\lambda_1^* - \lambda_1) f_1 g_1^*}{|f_1|^2 + |g_1|^2 + |h_1|^2}, \qquad \hat{v} = v + \frac{2\,\mathrm{i}(\lambda_1^* - \lambda_1) f_1 h_1^*}{|f_1|^2 + |g_1|^2 + |h_1|^2}, \tag{5}$$

where $\hat{\Psi} = (\hat{\psi}_1, \hat{\psi}_2, \hat{\psi}_3)^T$ *satisfies the Lax pair for the new potential* $(\hat{u}, \hat{v})$*,* $\lambda_1 \neq \lambda_1^*$ *and* I_3 *is the* 3×3 *identity matrix.*

Proof. With the condition that $(f_1, g_1, h_1)^T$ solves equations (3*a*) and (3*b*) with $\lambda = \lambda_1$, it can be verified that $(\hat{\Psi}, \hat{u}, \hat{v})$ also satisfies the Lax pair (3) by using MATHEMATICA or MAPLE (further details can be seen in [12]). □

From the knowledge of the DT, we only need to solve equations (3*a*) and (3*b*) with an initial potential $(u, v) = (u_0, v_0)$, and then obtain a series of explicit solutions to equations (2*a*) and (2*b*) in a recursive manner with the aid of symbolic computation [24, 25]. The iterative algorithm based on transformations (4) and (5) can be summarized as follows: (1) solve equations (3*a*) and (3*b*) with $u = u_0$ and $v = v_0$ for different spectral parameters $\lambda = \lambda_k$ $(1 \leqslant k \leqslant N)$, and obtain a set of linearly independent solutions $\{(f_k, g_k, h_k)^T\}_{k=1}^N$; (2) successive iteration of the DT N times yields the Nth-iterated potential and eigenfunction as follows:

$$u_N = u_0 + 2\,\mathrm{i} \sum_{k=1}^{N} \frac{(\lambda_k^* - \lambda_k) f_{k,\lambda_k} g_{k,\lambda_k}^*}{|f_{k,\lambda_k}|^2 + |g_{k,\lambda_k}|^2 + |h_{k,\lambda_k}|^2}, \tag{6}$$

$$v_N = v_0 + 2\,\mathrm{i} \sum_{k=1}^{N} \frac{(\lambda_k^* - \lambda_k) f_{k,\lambda_k} h_{k,\lambda_k}^*}{|f_{k,\lambda_k}|^2 + |g_{k,\lambda_k}|^2 + |h_{k,\lambda_k}|^2}, \tag{7}$$

$$\Psi_{k,\lambda_k} = (\lambda_k I_3 - S_{k-1}) \times \cdots \times (\lambda_k I_3 - S_1)\Psi_{1,\lambda_k}, \tag{8}$$

with

$$S_k = P_k J_k P_k^{-1}, \qquad J_k = \begin{pmatrix} \lambda_k & 0 & 0 \\ 0 & \lambda_k^* & 0 \\ 0 & 0 & \lambda_k^* \end{pmatrix}, \qquad P_k = \begin{pmatrix} f_{k,\lambda_k} & -g_{k,\lambda_k}^* & -h_{k,\lambda_k}^* \\ g_{k,\lambda_k} & f_{k,\lambda_k}^* & 0 \\ h_{k,\lambda_k} & 0 & f_{k,\lambda_k}^* \end{pmatrix},$$

where $\Psi_{k,\lambda_k} = \left(f_{k,\lambda_k}, g_{k,\lambda_k}, h_{k,\lambda_k}\right)^T$ $(2 \leqslant k \leqslant N)$, $\Psi_{1,\lambda_k} = (f_k, g_k, h_k)^T$ $(1 \leqslant k \leqslant N)$, (u_N, v_N) satisfies equations (2*a*) and (2*b*).

3. Triple Wronskian solutions

By setting $u_0 = v_0 = 0$ and after N-time iteration of the above DT-based algorithm, equations (2*a*) and (2*b*) are found to admit the following solutions:

$$u = -\frac{g}{f}, \qquad v = (-1)^{N-1}\frac{h}{f}, \tag{9}$$

where f, g and h are the complex functions which can be expressed as

$$f = \begin{vmatrix} F_{N\times N} & -G_{N\times N} & -H_{N\times N} \\ G_{N\times N}^* & F_{N\times N}^* & \mathbf{0} \\ H_{N\times N}^* & \mathbf{0} & F_{N\times N}^* \end{vmatrix}, \qquad g = 2\begin{vmatrix} F_{N\times(N+1)} & -G_{N\times(N-1)} & -H_{N\times N} \\ G_{N\times(N+1)}^* & F_{N\times(N-1)}^* & \mathbf{0} \\ H_{N\times(N+1)}^* & \mathbf{0} & F_{N\times N}^* \end{vmatrix},$$

$$h = 2\begin{vmatrix} F_{N\times(N+1)} & -G_{N\times N} & -H_{N\times(N-1)} \\ G_{N\times(N+1)}^* & F_{N\times N}^* & \mathbf{0} \\ H_{N\times(N+1)}^* & \mathbf{0} & F_{N\times(N-1)}^* \end{vmatrix}, \tag{10}$$

with $F_{N\times M}$, $G_{N\times M}$ and $H_{N\times M}$ $(M = N-1, N, N+1)$ as

$$F_{N\times M} = \begin{pmatrix} f_1 & f_1^{(1)} & \cdots & f_1^{(M-1)} \\ f_2 & f_2^{(1)} & \cdots & f_2^{(M-1)} \\ \vdots & \vdots & \ddots & \vdots \\ f_N & f_N^{(1)} & \cdots & f_N^{(M-1)} \end{pmatrix}, \qquad G_{N\times M} = \begin{pmatrix} g_1 & g_1^{(1)} & \cdots & g_1^{(M-1)} \\ g_2 & g_2^{(1)} & \cdots & g_2^{(M-1)} \\ \vdots & \vdots & \ddots & \vdots \\ g_N & g_N^{(1)} & \cdots & g_N^{(M-1)} \end{pmatrix},$$
$$H_{N\times M} = \begin{pmatrix} h_1 & h_1^{(1)} & \cdots & h_1^{(M-1)} \\ h_2 & h_2^{(1)} & \cdots & h_2^{(M-1)} \\ \vdots & \vdots & \ddots & \vdots \\ h_N & h_N^{(1)} & \cdots & h_N^{(M-1)} \end{pmatrix}. \tag{11}$$

Here $f_k^{(j)} = \partial^j f_k/\partial x^j$, $g_k^{(j)} = \partial^j g_k/\partial x^j$, $h_k^{(j)} = \partial^j h_k/\partial x^j$ $(k = 1, 2, \ldots, N;\ j = 1, 2, \ldots, M)$, and $\{(f_k, g_k, h_k)^T\}_{k=1}^N$ are the set of linearly independent solutions of the linear system

$$\begin{pmatrix} f \\ g \\ h \end{pmatrix}_x = \begin{pmatrix} -\mathrm{i}\lambda & 0 & 0 \\ 0 & \mathrm{i}\lambda & 0 \\ 0 & 0 & \mathrm{i}\lambda \end{pmatrix} \begin{pmatrix} f \\ g \\ h \end{pmatrix}, \qquad \begin{pmatrix} f \\ g \\ h \end{pmatrix}_t = \begin{pmatrix} -2\mathrm{i}\lambda^2 & 0 & 0 \\ 0 & 2\,\mathrm{i}\lambda^2 & 0 \\ 0 & 0 & 2\,\mathrm{i}\lambda^2 \end{pmatrix} \begin{pmatrix} f \\ g \\ h \end{pmatrix}. \tag{12}$$

Besides, the complex conjugates of u and v can be written as

$$u^* = -\frac{\bar{g}}{f}, \qquad v^* = (-1)^{N-1}\frac{\bar{h}}{f}, \tag{13}$$

where f has been presented as above and $\bar{g}$ and $\bar{h}$ are given by

$$\bar{g} = 2\begin{vmatrix} F_{N\times(N-1)} & -G_{N\times(N+1)} & -H_{N\times N} \\ G^*_{N\times(N-1)} & F^*_{N\times(N+1)} & \mathbf{0} \\ H^*_{N\times(N-1)} & \mathbf{0} & F^*_{N\times N} \end{vmatrix}, \qquad \bar{h} = 2\begin{vmatrix} F_{N\times(N-1)} & -G_{N\times N} & -H_{N\times(N+1)} \\ G^*_{N\times(N-1)} & F^*_{N\times N} & \mathbf{0} \\ H^*_{N\times(N-1)} & \mathbf{0} & F^*_{N\times(N+1)} \end{vmatrix}. \tag{14}$$

By extending the double Wronskian notation [21], we introduce the triple Wronskian notation for a determinant in the general form

$$\begin{aligned} &|\widehat{N-m_1}, N-j_1, \ldots, N-j_{m_1}; \widehat{N-m_2}, N-j_2, \ldots, N-j_{m_2}; \\ &\qquad \widehat{N-m_3}, N-j_3, \ldots, N-j_{m_3}|, \end{aligned} \tag{15}$$

where $\widehat{N-m_1} = (\phi, \phi^{(1)}, \ldots, \phi^{(N-m_1)})$, $\widehat{N-m_2} = (\varphi, \varphi^{(1)}, \ldots, \varphi^{(N-m_2)})$, $\widehat{N-m_3}$ $(\chi, \chi^{(1)}, \ldots, \chi^{(N-m_3)})$, $N-h_1 = \phi^{(N-h_1)}$, $N-h_2 = \varphi^{(N-h_2)}$, $N-h_3 = \chi^{(N-h_3)}$ $(j_l \leqslant h_l \leqslant j_{m_l}, l = 1, 2, 3)$, and $\phi = (f_1, \ldots, f_N;\ g_1^*, \ldots, g_N^*;\ h_1^*, \ldots, h_N^*)^T$, $\varphi = (-g_1, \ldots, -g_N;\ f_1^*, \ldots, f_N^*; 0, \ldots, 0)^T$, $\chi = (-h_1, \ldots, -h_N; 0, \ldots, 0;\ f_1^*, \ldots, f_N^*)^T$.

Following this new type of notation, the functions f, g, h, $\bar{g}$ and $\bar{h}$ can be written as

$$\begin{aligned} &f = |\widehat{N-1}; \widehat{N-1}; \widehat{N-1}|, \qquad g = 2|\widehat{N}; \widehat{N-2}; \widehat{N-1}|, \\ &h = 2|\widehat{N}; \widehat{N-1}; \widehat{N-2}|, \qquad \bar{g} = 2|\widehat{N-2}; \widehat{N}; \widehat{N-1}|, \\ &\bar{h} = 2|\widehat{N-2}; \widehat{N-1}; \widehat{N}|. \end{aligned} \tag{16}$$

According to the property that the derivative of most of the columns in (15) leads to a latter column and zero contribution, one can easily obtain various-order derivatives of f, g and h in the triple Wronskian form (see the appendix).

Substitution of (9) and (13) into equations (2*a*) and (2*b*) yields

$$f\left[\left(\mathrm{i}D_t + D_x^2\right)(g\cdot f)\right] - g\left[D_x^2(f\cdot f) - 2(g\bar{g} + h\bar{h})\right] = 0, \tag{17a}$$

$$f\left[\left(\mathrm{i}D_t + D_x^2\right)(h\cdot f)\right] - h\left[D_x^2(f\cdot f) - 2(g\bar{g} + h\bar{h})\right] = 0, \tag{17b}$$

where D_t, D_x and D_x^2 are the bilinear operators [18] defined by

$$D_x^m D_t^n (a\cdot b) \equiv \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^m \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^n a(x,t)b(x',t')\bigg|_{x'=x,t'=t}. \tag{18}$$

Before giving the proof that f, g, h, $\bar{g}$ and $\bar{h}$ satisfy equations (17*a*) and (17*b*), we first present the following lemmas [21, 22].

Lemma 3.1. *Suppose that Q is an $n\times(n-2)$ matrix and p, q, r and s are the n-dimensional column vectors. Then, we have*

$$|Q,p,q||Q,r,s| - |Q,p,r||Q,q,s| + |Q,p,s||Q,q,r| = 0. \tag{19}$$

Lemma 3.2. *For a matrix $\Pi = (\pi_{ij})_{n\times n} = [\Pi_1,\ldots,\Pi_n]$ and a column vector $\Upsilon = (r_1,\ldots,r_n)^T$, we have the relation*

$$\sum_{j=1}^{n} |\Pi_1,\ldots,\Pi_{j-1},\Upsilon\circ\Pi_j,\Pi_{j+1},\ldots,\Pi_n| = |\Pi|\sum_{j=1}^{n} r_j, \tag{20}$$

where $\Upsilon\circ\Pi_j = (r_1\pi_{1j}, r_2\pi_{2j},\ldots,r_n\pi_{nj})^T$.

As a direct result of lemma 3.2, two useful triple Wronskian identities are obtained as follows:

$$\begin{aligned}
&|\widehat{N_1-1};\widehat{N_2-1};\widehat{N_3-1}|\sum_{k=1}^{N}(-\mathrm{i}\lambda_k - 2\,\mathrm{i}\lambda_k^*)\\
&\quad = |\widehat{N_1-2},N_1;\widehat{N_2-1};\widehat{N_3-1}| - |\widehat{N_1-1};\widehat{N_2-2},N_2;\widehat{N_3-1}|\\
&\qquad - |\widehat{N_1-1};\widehat{N_2-1};\widehat{N_3-2},N_3|,
\end{aligned} \tag{21}$$

$$\begin{aligned}
&|\widehat{N_1-1};\widehat{N_2-1};\widehat{N_3-1}|\left[\sum_{k=1}^{N}(-\mathrm{i}\lambda_k - 2\,\mathrm{i}\lambda_k^*)\right]^2\\
&\quad = |\widehat{N_1-3},N_1-1,N_1;\widehat{N_2-1};\widehat{N_3-1}| + |\widehat{N_1-2},N_1+1;\widehat{N_2-1};\widehat{N_3-1}|\\
&\qquad + |\widehat{N_1-1};\widehat{N_2-3},N_2-1,N_2;\widehat{N_3-1}| - 2|\widehat{N_1-2},N_1;\widehat{N_2-2},N_2;\widehat{N_3-1}|\\
&\qquad + |\widehat{N_1-1};\widehat{N_2-2},N_2+1;\widehat{N_3-1}| - 2|\widehat{N_1-2},N_1;\widehat{N_2-1};\widehat{N_3-2},N_3|\\
&\qquad + |\widehat{N_1-1};\widehat{N_2-1};\widehat{N_3-3},N_3-1,N_3| + 2|\widehat{N_1-1};\widehat{N_2-2},N_2;\widehat{N_3-2},N_3|\\
&\qquad + |\widehat{N_1-1};\widehat{N_2-1};\widehat{N_3-2},N_3+1|,
\end{aligned} \tag{22}$$

where N_1, N_2 and N_3 represent the arbitrary natural numbers.

Lemma 3.3. *Let $W = (\widehat{N-1};\widehat{N-3};\widehat{N-3})$, $\Phi_1 = (\phi,\varphi,\chi)$ and $\Phi_2 = (\phi^{(1)},\ldots,\phi^{(N)};\varphi^{(1)},\ldots,\varphi^{(N-2)};\chi^{(1)},\ldots,\chi^{(N-2)})$. The determinants of the following two matrices*

$$X_1 = \begin{pmatrix} W & \mathbf{0} & \mathbf{0} & \chi^{(N-2)} & \chi^{(N-1)} & \chi^{(N)} & \varphi^{(N-2)} & \varphi^{(N-1)} \\ \mathbf{0} & \Phi_1 & \Phi_2 & \chi^{(N-1)} & \chi^{(N)} & \chi^{(N+1)} & \varphi^{(N-1)} & \varphi^{(N)} \end{pmatrix}, \tag{23}$$

$$X_2 = \begin{pmatrix} W & \mathbf{0} & \mathbf{0} & \varphi^{(N-2)} & \varphi^{(N-1)} & \varphi^{(N)} & \chi^{(N-2)} & \chi^{(N-1)} \\ \mathbf{0} & \Phi_1 & \Phi_2 & \varphi^{(N-1)} & \varphi^{(N)} & \varphi^{(N+1)} & \chi^{(N-1)} & \chi^{(N)} \end{pmatrix}, \tag{24}$$

are equal to zero.

Proof. From the conditions that $f_k^{(j)} = -\mathrm{i}\lambda_k f_k^{(j-1)}$, $g_k^{(j)} = \mathrm{i}\lambda_k g_k^{(j-1)}$ and $h_k^{(j)} = \mathrm{i}\lambda_k h_k^{(j-1)}$, we have

$$\Xi W = (\phi^{(1)}, \ldots, \phi^{(N)}; -\varphi^{(1)}, \ldots, -\varphi^{(N-2)}; -\chi^{(1)}, \ldots, -\chi^{(N-2)}), \tag{25}$$

$$\Xi(\chi^{(N-2)}, \chi^{(N-1)}, \chi^{(N)}, \varphi^{(N-2)}, \varphi^{(N-1)}) = -(\chi^{(N-1)}, \chi^{(N)}, \chi^{(N+1)}, \varphi^{(N-1)}, \varphi^{(N)}), \tag{26}$$

$$\Xi(\varphi^{(N-2)}, \varphi^{(N-1)}, \varphi^{(N)}, \chi^{(N-2)}, \chi^{(N-1)}) = -(\varphi^{(N-1)}, \varphi^{(N)}, \varphi^{(N+1)}, \chi^{(N-1)}, \chi^{(N)}), \tag{27}$$

where $\Xi = \mathrm{diag}(-\mathrm{i}\lambda_1, \ldots, -\mathrm{i}\lambda_N; -\mathrm{i}\lambda_1^*, \ldots, -\mathrm{i}\lambda_N^*; -\mathrm{i}\lambda_1^*, \ldots, -\mathrm{i}\lambda_N^*)$. Through suitable row and column operations, X_1 and X_2 can be transformed into

$$X_1' = YX_1Z = \begin{pmatrix} W & \mathbf{0} & \mathbf{0} & \chi^{(N-2)} & \chi^{(N-1)} & \chi^{(N)} & \varphi^{(N-2)} & \varphi^{(N-1)} \\ \mathbf{0} & \Phi_1 & \Phi_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}, \tag{28}$$

$$X_2' = YX_2Z = \begin{pmatrix} W & \mathbf{0} & \mathbf{0} & \varphi^{(N-2)} & \varphi^{(N-1)} & \varphi^{(N)} & \chi^{(N-2)} & \chi^{(N-1)} \\ \mathbf{0} & \Phi_1 & \Phi_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}, \tag{29}$$

with

$$Y = \begin{pmatrix} I_{3N} & \mathbf{0} \\ \Xi & I_{3N} \end{pmatrix}, \qquad Z = \begin{pmatrix} I_N & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & I_{2N-4} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -I_N & \mathbf{0} & \mathbf{0} & I_N & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & I_{2N-4} & \mathbf{0} & \mathbf{0} & I_{2N-4} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & I_5 \end{pmatrix}.$$

Applying the Laplace expansion in terms of $3N \times 3N$ minors to $|X_1'|$ and $|X_2'|$, we have that $|X_1'| = 0$ and $|X_2'| = 0$, that is, $|X_1| = 0$ and $|X_2| = 0$. □

Theorem 3.4. *The coupled NLS equations (2a) and (2b) admit the triple Wronskian solutions (9) together with (10) and (11), provided that $\{(f_k, g_k, h_k)\}_{k=1}^N$ satisfy the linear system (12).*

Proof. By using the triple Wronskian notations of $f_x, g_x, h_x, f_t, g_t, h_t, f_{xx}, g_{xx}, h_{xx}$ and the identities (21) and (22), we obtain the following:

$$\begin{aligned} D_x^2(f\cdot f) - 2(g\bar{g} + h\bar{h}) = {} & 4(|\widehat{N-1}; \widehat{N-1}; \widehat{N-1}||\widehat{N-2, N}; \widehat{N-2, N}; \widehat{N-1}| \\ & - |\widehat{N}; \widehat{N-2}; \widehat{N-1}||\widehat{N-2}; \widehat{N}; \widehat{N-1}| \\ & - |\widehat{N-2, N}; \widehat{N-1}; \widehat{N-1}||\widehat{N-1}; \widehat{N-2, N}; \widehat{N-1}|) \\ & - 4(|\widehat{N}; \widehat{N-1}; \widehat{N-2}||\widehat{N-2}; \widehat{N-1}; \widehat{N}| \\ & + |\widehat{N-2, N}; \widehat{N-1}; \widehat{N-1}||\widehat{N-1}; \widehat{N-1}; \widehat{N-2, N}| \\ & - |\widehat{N-1}; \widehat{N-1}; \widehat{N-1}||\widehat{N-2, N}; \widehat{N-1}; \widehat{N-2, N}|), \end{aligned} \tag{30}$$

$$
\begin{aligned}
\mathrm{i}D_t(g\cdot f)+D_x^2(g\cdot f) = {}& 8(|\widehat{N-1};\,\widehat{N-1};\,\widehat{N-1}||\widehat{N-2},N,N+1;\,\widehat{N-2};\,\widehat{N-1}| \\
&+|\widehat{N-2},N+1;\,\widehat{N-1};\,\widehat{N-1}||\widehat{N};\,\widehat{N-2};\,\widehat{N-1}| \\
&-|\widehat{N-1},N+1;\,\widehat{N-2};\,\widehat{N-1}||\widehat{N-2},N;\,\widehat{N-1};\,\widehat{N-1}|) \\
&+8(|\widehat{N-1};\,\widehat{N-1};\,\widehat{N-1}||\widehat{N};\,\widehat{N-3},N;\,\widehat{N-1}| \\
&+|\widehat{N-1};\,\widehat{N-3},N-1,N;\,\widehat{N-1}||\widehat{N};\,\widehat{N-2};\,\widehat{N-1}| \\
&-|\widehat{N};\,\widehat{N-3},N-1;\,\widehat{N-1}||\widehat{N-1};\,\widehat{N-2},N;\,\widehat{N-1}|) \\
&+8(|\widehat{N-1};\,\widehat{N-1};\,\widehat{N-1}||\widehat{N};\,\widehat{N-3},N-1;\,\widehat{N-2},N| \\
&+|\widehat{N-1};\,\widehat{N-3},N-1;\,\widehat{N}||\widehat{N};\,\widehat{N-1};\,\widehat{N-2}| \\
&-|\widehat{N};\,\widehat{N-3},N-1;\,\widehat{N-1}||\widehat{N-1};\,\widehat{N-1};\,\widehat{N-2},N|) \\
&-8(|\widehat{N-1};\,\widehat{N-2};\,\widehat{N}||\widehat{N};\,\widehat{N-2},N;\,\widehat{N-2}| \\
&+|\widehat{N-1};\,\widehat{N-2},N;\,\widehat{N-1}||\widehat{N};\,\widehat{N-2};\,\widehat{N-2},N| \\
&-|\widehat{N-1};\,\widehat{N-2},N;\,\widehat{N-2},N||\widehat{N};\,\widehat{N-2};\,\widehat{N-1}|) \\
&+8(|\widehat{N-1};\,\widehat{N-1};\,\widehat{N-1}||\widehat{N};\,\widehat{N-2};\,\widehat{N-2},N+1| \\
&+|\widehat{N};\,\widehat{N-2},N;\,\widehat{N-2}||\widehat{N-1};\,\widehat{N-2};\,\widehat{N}| \\
&+|\widehat{N-1};\,\widehat{N-1};\,\widehat{N-3},N-1,N||\widehat{N};\,\widehat{N-2};\,\widehat{N-1}| \\
&-|\widehat{N-1};\,\widehat{N-3},N-1;\,\widehat{N}||\widehat{N};\,\widehat{N-1};\,\widehat{N-2}| \\
&-|\widehat{N-1};\,\widehat{N-1};\,\widehat{N-2},N||\widehat{N};\,\widehat{N-2};\,\widehat{N-2},N|),
\end{aligned}
\tag{31}
$$

$$
\begin{aligned}
\mathrm{i}\,D_t(h\cdot f)+D_x^2(h\cdot f) = {}& 8(|\widehat{N-1};\,\widehat{N-1};\,\widehat{N-1}||\widehat{N-2},N,N+1;\,\widehat{N-1};\,\widehat{N-2}| \\
&+|\widehat{N-2},N+1;\,\widehat{N-1};\,\widehat{N-1}||\widehat{N};\,\widehat{N-1};\,\widehat{N-2}| \\
&-|\widehat{N-1},N+1;\,\widehat{N-1};\,\widehat{N-2}||\widehat{N-2},N;\,\widehat{N-1};\,\widehat{N-1}|) \\
&+8(|\widehat{N-1};\,\widehat{N-1};\,\widehat{N-1}||\widehat{N};\,\widehat{N-1};\,\widehat{N-3},N| \\
&+|\widehat{N-1};\,\widehat{N-1};\,\widehat{N-3},N-1,N||\widehat{N};\,\widehat{N-1};\,\widehat{N-2}| \\
&-|\widehat{N};\,\widehat{N-1};\,\widehat{N-3},N-1||\widehat{N-1};\,\widehat{N-1};\,\widehat{N-2},N|) \\
&+8(|\widehat{N-1};\,\widehat{N-1};\,\widehat{N-1}||\widehat{N};\,\widehat{N-2},N;\,\widehat{N-3},N-1| \\
&+|\widehat{N-1};\,\widehat{N};\,\widehat{N-3},N-1||\widehat{N};\,\widehat{N-2};\,\widehat{N-1}| \\
&-|\widehat{N};\,\widehat{N-1};\,\widehat{N-3},N-1||\widehat{N-1};\,\widehat{N-2},N;\,\widehat{N-1}|) \\
&-8(|\widehat{N-1};\,\widehat{N};\,\widehat{N-2}||\widehat{N};\,\widehat{N-2};\,\widehat{N-2},N| \\
&+|\widehat{N-1};\,\widehat{N-1};\,\widehat{N-2},N||\widehat{N};\,\widehat{N-2},N;\,\widehat{N-2}| \\
&-|\widehat{N-1};\,\widehat{N-2},N;\,\widehat{N-2},N||\widehat{N};\,\widehat{N-1};\,\widehat{N-2}|) \\
&+8(|\widehat{N-1};\,\widehat{N-1};\,\widehat{N-1}||\widehat{N};\,\widehat{N-2},N+1;\,\widehat{N-2}| \\
&-|\widehat{N};\,\widehat{N-2};\,\widehat{N-2},N||\widehat{N-1};\,\widehat{N};\,\widehat{N-2}| \\
&+|\widehat{N-1};\,\widehat{N-3},N-1,N;\,\widehat{N-1}||\widehat{N};\,\widehat{N-1};\,\widehat{N-2}| \\
&+|\widehat{N-1};\,\widehat{N};\,\widehat{N-3},N-1||\widehat{N};\,\widehat{N-2};\,\widehat{N-1}| \\
&-|\widehat{N-1};\,\widehat{N-2},N;\,\widehat{N-1}||\widehat{N};\,\widehat{N-2},N;\,\widehat{N-2}|).
\end{aligned}
\tag{32}
$$

Via lemma 3.1, one can check that those two terms on the right-hand side of equation (30) and the first four terms on the right-hand sides of both equations (31) and (32) are identically equal to zero. By virtue of lemma 3.3, the expansion of $|X_1|$ (or $|X_2|$) in terms of $3N \times 3N$ minors indicates that the last term on the right-hand side of equation (31) (or equation (32)) is also equivalent to zero. Therefore, we draw the conclusion that $f, g, h, \bar{g}, \bar{h}$ satisfy equations (17*a*) and (17*b*), or equivalently, u and v given by (9) together with (10) and (11) solve equations (2*a*) and (2*b*). □

Remarks. *Since $u = -g/f$, $v = (-1)^{N-1}h/f$, $u^* = -\bar{g}/f$ and $v^* = (-1)^{N-1}\bar{h}/f$ have been proved to satisfy equations (17a) and (17b), we immediately have the following relations:*

$$\frac{\bar{g}}{f} = \frac{g^*}{f^*}, \qquad \frac{\bar{h}}{f} = \frac{h^*}{f^*}, \tag{33}$$

where f^*, g^* and h^* correspond to the complex conjugates of f, g and h, respectively. On the other hand, the determinantal expansion shows that $f \times \det(F_{N\times N})$ is a real function, i.e. $f \times \det(F_{N\times N}) = f^* \times \det(F^*_{N\times N})$, which implies that g^* and h^* can be expressed as

$$g^* = \frac{\det(F_{N\times N})}{\det(F^*_{N\times N})}\,\bar{g}, \qquad h^* = \frac{\det(F_{N\times N})}{\det(F^*_{N\times N})}\,\bar{h}. \tag{34}$$

4. Bright *N*-soliton solutions and asymptotic analysis

In this section, we find that the bright N-soliton solutions to equations (2*a*) and (2*b*) can be represented in terms of the triple Wronskian by solving the linear system (12). Then, we are devoted to the algebraic properties of the triple Wronskian and the procedure in deriving the asymptotic expressions of the bright N-soliton solutions with arbitrary N. For convenience, we define the notations $[n] := \{1, 2, \ldots, n\}$, $[k, n] := \{k, k+1, \ldots, n\}$ and use $|\cdot|$ to denote the number of elements of a set.

For different spectral parameters λ_k ($1 \leqslant k \leqslant N$), the general solutions of the linear system (12) can be obtained as follows:

$$(f_k, g_k, h_k) = (\alpha_k\, \mathrm{e}^{\theta_k}, \beta_k\, \mathrm{e}^{-\theta_k}, \gamma_k\, \mathrm{e}^{-\theta_k}), \tag{35}$$

where the phase $\theta_k = -\mathrm{i}\lambda_k x - 2\,\mathrm{i}\lambda_k^2 t$, α_k, β_k and γ_k are arbitrary complex constants.

With $N = 1$, we can express u and v as

$$u = \frac{\beta_1^*(\mathrm{i}\lambda_1^* - \mathrm{i}\lambda_1)|\alpha_1|}{\alpha_1^*\sqrt{|\beta_1|^2 + |\gamma_1|^2}}\,\mathrm{sech}(K_1 x - \Omega_1 t + \delta_1)\,\mathrm{e}^{-\mathrm{i}(\lambda_1+\lambda_1^*)x - 2\mathrm{i}(\lambda_1^2+\lambda_1^{*2})t}, \tag{36a}$$

$$v = \frac{\gamma_1^*(\mathrm{i}\lambda_1^* - \mathrm{i}\lambda_1)|\alpha_1|}{\alpha_1^*\sqrt{|\beta_1|^2 + |\gamma_1|^2}}\,\mathrm{sech}(K_1 x - \Omega_1 t + \delta_1)\,\mathrm{e}^{-\mathrm{i}(\lambda_1+\lambda_1^*)x - 2\mathrm{i}(\lambda_1^2+\lambda_1^{*2})t}, \tag{36b}$$

which are called the two-component bright one-soliton solutions, where α_1, β_1 and γ_1 are the arbitrary nonzero complex constants, $\lambda_1 \neq \lambda_1^*$, $K_1 = \mathrm{i}\lambda_1^* - \mathrm{i}\lambda_1$ is the wave number, $\Omega_1 = 2\,\mathrm{i}\left(\lambda_1^2 - \lambda_1^{*2}\right)$ is the frequency and $\delta_1 = -\frac{1}{2}\ln\left(\left|\frac{\beta_1}{\alpha_1}\right|^2 + \left|\frac{\gamma_1}{\alpha_1}\right|^2\right)$ is the initial phase. Apart from the constant δ_1, the bright one-soliton solutions are characterized by the soliton amplitudes and soliton velocities which are, respectively, given by

$$A_u = \frac{\left|\frac{\beta_1}{\alpha_1}\right|\left|\mathrm{i}\lambda_1 - \mathrm{i}\lambda_1^*\right|}{\sqrt{\left|\frac{\beta_1}{\alpha_1}\right|^2 + \left|\frac{\gamma_1}{\alpha_1}\right|^2}}, \qquad A_v = \frac{\left|\frac{\gamma_1}{\alpha_1}\right|\left|\mathrm{i}\lambda_1 - \mathrm{i}\lambda_1^*\right|}{\sqrt{\left|\frac{\beta_1}{\alpha_1}\right|^2 + \left|\frac{\gamma_1}{\alpha_1}\right|^2}}, \qquad V_u = V_v = -2(\lambda_1 + \lambda_1^*), \tag{37}$$

which means that solutions (36*a*) and (36*b*) are actually characterized by the values of $\frac{\beta_1}{\alpha_1}$, $\frac{\gamma_1}{\alpha_1}$ and λ_1.

For $N \geqslant 2$, u and v given by (9) together with (10) and (11) can describe the collision dynamics of two-component bright N-soliton solutions. In order to avoid the singular and reducible cases, we require that $\alpha_k \neq 0$, $(\beta_k, \gamma_k) \neq (0,0)$, $\lambda_k \neq 0$, $\lambda_k \neq \lambda_k^*$ and $\mathrm{Re}(\lambda_k) \neq \mathrm{Re}(\lambda_l)$, where $1 \leqslant k < l \leqslant N$. It can be found that the characterization of the bright N-soliton solutions depends on the values of $3N$ complex parameters, as stated in the following proposition.

Proposition 4.1. *The bright N-soliton solutions to the coupled NLS equations (2a) and (2b) are characterized by the set of parameters* $\left\{\left(\frac{\beta_k}{\alpha_k}, \frac{\gamma_k}{\alpha_k}, \lambda_k\right)\right\}_{k=1}^N$, *where* $\alpha_k \neq 0$, $(\beta_k, \gamma_k) \neq (0,0)$, $\lambda_k \neq 0$, $\lambda_k \neq \lambda_k^*$ *and* $\mathrm{Re}(\lambda_k) \neq \mathrm{Re}(\lambda_l)$ $(1 \leqslant k < l \leqslant N)$.

Proof. First, we write the functions f, g and h in the following form:

$$f = \begin{vmatrix} A\Theta_1\Lambda_{1,N} & -B\Theta_2\Lambda_{2,N} & -C\Theta_2\Lambda_{2,N} \\ B^*\Theta_2^*\Lambda_{2,N}^* & A^*\Theta_1^*\Lambda_{1,N}^* & \mathbf{0} \\ C^*\Theta_2^*\Lambda_{2,N}^* & \mathbf{0} & A^*\Theta_1^*\Lambda_{1,N}^* \end{vmatrix}, \tag{38}$$

$$g = 2\begin{vmatrix} A\Theta_1\Lambda_{1,N+1} & -B\Theta_2\Lambda_{2,N-1} & -C\Theta_2\Lambda_{2,N} \\ B^*\Theta_2^*\Lambda_{2,N+1}^* & A^*\Theta_1^*\Lambda_{1,N-1}^* & \mathbf{0} \\ C^*\Theta_2^*\Lambda_{2,N+1}^* & \mathbf{0} & A^*\Theta_1^*\Lambda_{1,N}^* \end{vmatrix}, \tag{39}$$

$$h = 2\begin{vmatrix} A\Theta_1\Lambda_{1,N+1} & -B\Theta_2\Lambda_{2,N} & -C\Theta_2\Lambda_{2,N-1} \\ B^*\Theta_2^*\Lambda_{2,N+1}^* & A^*\Theta_1^*\Lambda_{1,N}^* & \mathbf{0} \\ C^*\Theta_2^*\Lambda_{2,N+1}^* & \mathbf{0} & A^*\Theta_1^*\Lambda_{1,N-1}^* \end{vmatrix}, \tag{40}$$

where $A = \mathrm{diag}(\alpha_1, \ldots, \alpha_N)$, $B = \mathrm{diag}(\beta_1, \ldots, \beta_N)$, $C = \mathrm{diag}(\gamma_1, \ldots, \gamma_N)$, $\Theta_1 = (\mathrm{e}^{\theta_1}, \ldots, \mathrm{e}^{\theta_N})$, $\Theta_2 = (\mathrm{e}^{-\theta_1}, \ldots, \mathrm{e}^{-\theta_N})$, $\Lambda_{1,M}$ and $\Lambda_{2,M}$ $(M = N-1, N, N+1)$ are expressed as

$$\Lambda_{1,M} = \begin{pmatrix} 1 & -\mathrm{i}\lambda_1 & \cdots & (-\mathrm{i}\lambda_1)^{M-1} \\ 1 & -\mathrm{i}\lambda_2 & \cdots & (-\mathrm{i}\lambda_2)^{M-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & -\mathrm{i}\lambda_N & \cdots & (-\mathrm{i}\lambda_N)^{M-1} \end{pmatrix}, \qquad \Lambda_{2,M} = \begin{pmatrix} 1 & \mathrm{i}\lambda_1 & \cdots & (\mathrm{i}\lambda_1)^{M-1} \\ 1 & \mathrm{i}\lambda_2 & \cdots & (\mathrm{i}\lambda_2)^{M-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \mathrm{i}\lambda_N & \cdots & (\mathrm{i}\lambda_N)^{M-1} \end{pmatrix}. \tag{41}$$

Simultaneous multiplication of f, g and h by $\Gamma = \left|\mathrm{diag}\left(\alpha_1^{-1}, \ldots, \alpha_N^{-1}; \alpha_1^{*-1}, \ldots, \alpha_N^{*-1}; \alpha_1^{*-1}, \ldots, \alpha_N^{*-1}\right)\right|$ gives rise to

$$f' = \Gamma f = \begin{vmatrix} \Theta_1\Lambda_{1,N} & -B'\Theta_2\Lambda_{2,N} & -C'\Theta_2\Lambda_{2,N} \\ B'^*\Theta_2^*\Lambda_{2,N}^* & \Theta_1^*\Lambda_{1,N}^* & \mathbf{0} \\ C'^*\Theta_2^*\Lambda_{2,N}^* & \mathbf{0} & \Theta_1^*\Lambda_{1,N}^* \end{vmatrix}, \tag{42}$$

$$g' = \Gamma g = 2\begin{vmatrix} \Theta_1\Lambda_{1,N+1} & -B'\Theta_2\Lambda_{2,N-1} & -C'\Theta_2\Lambda_{2,N} \\ B'^*\Theta_2^*\Lambda_{2,N+1}^* & \Theta_1^*\Lambda_{1,N-1}^* & \mathbf{0} \\ C'^*\Theta_2^*\Lambda_{2,N+1}^* & \mathbf{0} & \Theta_1^*\Lambda_{1,N}^* \end{vmatrix}, \tag{43}$$

$$h' = \Gamma h = 2\begin{vmatrix} \Theta_1\Lambda_{1,N+1} & -B'\Theta_2\Lambda_{2,N} & -C'\Theta_2\Lambda_{2,N-1} \\ B'^*\Theta_2^*\Lambda_{2,N+1}^* & \Theta_1^*\Lambda_{1,N}^* & \mathbf{0} \\ C'^*\Theta_2^*\Lambda_{2,N+1}^* & \mathbf{0} & \Theta_1^*\Lambda_{1,N-1}^* \end{vmatrix}, \tag{44}$$

where $B' = \mathrm{diag}\left(\frac{\beta_1}{\alpha_1}, \ldots, \frac{\beta_N}{\alpha_N}\right)$ and $C' = \mathrm{diag}\left(\frac{\gamma_1}{\alpha_1}, \ldots, \frac{\gamma_N}{\alpha_N}\right)$. Note that $u = g/f = g'/f'$ and $v = h/f = h'/f'$; then the characteristics of the bright N-soliton solutions is completely determined by the parameter space $\left\{\left(\frac{\beta_k}{\alpha_k}, \frac{\gamma_k}{\alpha_k}, \lambda_k\right)\right\}_{k=1}^{N}$. □

Without loss of generality, we take $\alpha_k = 1$ $(k = 1, \ldots, N)$ and define the set $\mathcal{S}_N = \{(\beta_k, \gamma_k, \lambda_k)\}_{k=1}^{N}$ as the parameter space of the bright N-soliton solutions to the coupled NLS equations (2*a*) and (2*b*). In the following, we first present two important lemmas which will be used to analyze the asymptotic behavior of the generic bright N-soliton solutions to equations (2*a*) and (2*b*).

Lemma 4.2. *Assume that* $\xi_k = \mathrm{Re}(\theta_k) = (\theta_k + \theta_k^*)/2 = \frac{1}{2}\mu_k(x - 2\nu_k t)$, *where* $\mu_k = \mathrm{i}(\lambda_k^* - \lambda_k)$, $\nu_k = -\lambda_k - \lambda_k^*$ *and* $\nu_1 < \nu_2 < \cdots < \nu_N$. *If* $\xi_j \sim 0$ *for some* $1 \leqslant j \leqslant N$, *the limits of* ξ_k $(k = 1, \ldots, N$ *and* $k \neq j)$ *as* $t \to \mp\infty$ *are given as follows:*

(a) If $\mu_k > 0$, $\xi_k \sim -\infty$ $(k = 1, \ldots, j-1)$ *and* $\xi_k \sim +\infty$ $(k = j+1, \ldots, N)$ *as* $t \to -\infty$; $\xi_k \sim +\infty$ $(k = 1, \ldots, j-1)$ *and* $\xi_k \sim -\infty$ $(k = j+1, \ldots, N)$ *as* $t \to +\infty$.
(b) If $\mu_k < 0$, $\xi_k \sim +\infty$ $(k = 1, \ldots, j-1)$ *and* $\xi_k \sim -\infty$ $(k = j+1, \ldots, N)$ *as* $t \to -\infty$; $\xi_k \sim -\infty$ $(k = 1, \ldots, j-1)$ *and* $\xi_k \sim +\infty$ $(k = j+1, \ldots, N)$ *as* $t \to +\infty$.

Proof. The above results directly follow from the relation $\mu_j\xi_k = \mu_k\,\xi_j + \mu_j\mu_k(\nu_j - \nu_k)\,t$. □

Lemma 4.3. *Let* $\vartheta_{\mathrm{I}} = \sum_{k=1}^{N}\left(a_k\theta_k + b_k\theta_k^*\right)$ *and* $\vartheta_{\mathrm{II}} = \sum_{k=1}^{N}\left(c_k\theta_k + d_k\theta_k^*\right)$, *respectively, denote the linear phase combinations actually appearing in the expansions of the determinants* f *and* g *(or* h*). Assume that* $(\beta_k, \gamma_k) \neq (0, 0)$, $\lambda_k \neq 0$, $\lambda_k \neq \lambda_k^*$ *and* $\mathrm{Re}(\lambda_k) \neq \mathrm{Re}(\lambda_l)$, *where* $1 \leqslant k < l \leqslant N$. *Then, the coefficients* a_k, b_k, c_k *and* d_k $(1 \leqslant k \leqslant N)$ *are subject to the following properties:*

(i) $a_k, c_k \in \{-1, 1\}$ *and* $b_k, d_k \in \{0, 2\}$.
(ii) $|\{a_k|a_k = -1\}| = |\{b_k|b_k = 0\}|$ *and* $|\{a_k|a_k = 1\}| = |\{b_k|b_k = 2\}|$.
(iii) $|\{d_k|d_k = 0\}| - |\{c_k|c_k = -1\}| = 1$ *and* $|\{c_k|c_k = 1\}| - |\{d_k|d_k = 2\}| = 1$.

Proof. Let A_{I} and A_{II}, respectively, represent the matrices associated with the determinants f and g (or h), and let $\Delta_{\mathcal{I}}^{\mathcal{J}}(A_{\mathrm{I}})$ and $\Delta_{\mathcal{I}}^{\mathcal{J}}(A_{\mathrm{II}})$ be the minors of f and g (or h) with rows $\mathcal{I}$ and columns $\mathcal{J}$, where both $\mathcal{I}$, $\mathcal{J}$ are the subsets of $[3N]$ and $|\mathcal{I}| = |\mathcal{J}|$. We define $\{\mathcal{I}_{k+1}\}_{k=0}^{N}$ and $\{\mathcal{J}_{k+1}\}_{k=0}^{N}$ as two sequences of subsets, where $\mathcal{I}_1 = [N]$, $\mathcal{J}_1 = \{j_{11}, j_{12}, \ldots, j_{1N}\}$, $\mathcal{I}_{k+1} = \{N+k, 2N+k\}$ and $\mathcal{J}_{k+1} = \{j_{k+1,1}, j_{k+1,2}\}$ for $1 \leqslant k \leqslant N$, and the sequence of subsets $\{\mathcal{J}_{k+1}\}_{k=0}^{N}$ form a disjoint partition of $[3N]$, that is

$$\bigcup_{k=0}^{N}\mathcal{J}_{k+1} = [3N] \quad \text{and} \quad \mathcal{J}_{k+1} \cap \mathcal{J}_{l+1} = \varnothing \quad (0 \leqslant k < l \leqslant N).$$

(i) Because $(\beta_k, \gamma_k) \neq (0, 0)$, and e^{θ_k} and $\mathrm{e}^{-\theta_k}$ $(1 \leqslant k \leqslant N)$ only appear in the kth row of A_{I} and A_{II}, the determinantal properties show that the coefficients a_k and c_k must be equal to 1 or -1. On the other hand, we note that $\mathrm{e}^{\theta_k^*}$ and $\mathrm{e}^{-\theta_k^*}$ are contained in the $(N+k)$th and $(2N+k)$th rows of A_{I} and A_{II}. Therefore, the coefficients b_k and d_k must belong to $\{0, 2\}$.

(ii) For the phase combination ϑ_{I} associated with the disjoint partition $\{\mathcal{J}_{k+1}\}_{k=0}^{N}$ of $[3N]$ such that $\Delta_{\mathcal{I}_{k+1}}^{\mathcal{J}_{k+1}}(A_{\mathrm{I}}) \neq 0$ $(0 \leqslant k \leqslant N)$, we assume that $r_{\mathrm{I}} = |\{a_k|a_k = -1\}|$ and $s_{\mathrm{I}} = |\{b_k|b_k = 0\}|$. According to part (i) of this lemma, it is easy to know that $|\{a_k|a_k = 1\}| = N - r_{\mathrm{I}}$ and $|\{b_k|b_k = 2\}| = N - s_{\mathrm{I}}$.

   (a) If $r_{\mathrm{I}} = 0$, then $\mathcal{J}_1 = [N]$ and $\mathcal{J}_1 \cap \mathcal{J}_{k+1} = \varnothing$ $(1 \leqslant k \leqslant N)$, which means that $a_k = 1$ and $b_k = 2$ for all $1 \leqslant k \leqslant N$, that is to say, $s_{\mathrm{I}} = r_{\mathrm{I}} = 0$.

(b) If $1 \leqslant r_{\mathrm{I}} \leqslant N$, then $\mathcal{J}_1 \cap [N+1, 3N] = \left\{j_{1,l_1}, \ldots, j_{1,l_{r_{\mathrm{I}}}}\right\}$ with $1 \leqslant l_1 < \cdots < l_{r_{\mathrm{I}}} \leqslant N$. Note that $\Delta_{\mathcal{I}_{k+1}}^{\mathcal{J}_{k+1}}(A_{\mathrm{I}}) = 0$ if $|\mathcal{J}_{k+1} \cap [N]| = 2$ $(1 \leqslant k \leqslant N)$. Accordingly, there exist r_{I} and only r_{I} subsets $\{\mathcal{J}_{k_n+1}\}_{n=1}^{r_{\mathrm{I}}}$ of $[3N]$ such that $|\mathcal{J}_{k_n+1} \cap [N]| = 1$, where $1 \leqslant k_1 < \cdots < k_{r_{\mathrm{I}}} \leqslant N$. Thus, $b_{k_n} = 0$ for all $1 \leqslant n \leqslant r_{\mathrm{I}}$, which implies that $s_{\mathrm{I}} = r_{\mathrm{I}}$.

(iii) Any phase combination ϑ_{II} actually appearing in the determinant g (or h) must be associated with a disjoint partition $\{\mathcal{J}_{k+1}\}_{k=0}^{N}$ of $[3N]$ with $\Delta_{\mathcal{I}_{k+1}}^{\mathcal{J}_{k+1}}(A_{\mathrm{II}}) \neq 0$ $(0 \leqslant k \leqslant N)$. As for this phase combination, we take $r_{\mathrm{II}} = |\{c_k | c_k = -1\}|$, $s_{\mathrm{II}} = |\{d_k | d_k = 0\}|$, $|\{c_k | c_k = 1\}| = N - r_{\mathrm{II}}$ and $|\{d_k | d_k = 2\}| = N - s_{\mathrm{II}}$.

(a) If $r_{\mathrm{II}} = 0$, then $\mathcal{J}_1 \subset [N+1]$. In this case, there is one and only one $k^* \in [N]$ such that $|\mathcal{J}_{k^*} \cap [N+1]| = 1$ and $\mathcal{J}_k \cap [N+1] = \varnothing$ for $k \neq k^*$. That is to say, $c_k = 1$ for all $1 \leqslant k \leqslant N$, $d_{k^*} = 0$ and $d_k = 2$ for $k \neq k^*$. Therefore, we know that $r_{\mathrm{II}} = 0$ and $s_{\mathrm{II}} = 1$.

(b) If $1 \leqslant r_{\mathrm{II}} < N$, then $\mathcal{J}_1 \cap [N+2, 3N] = \left\{j_{1,l_1}, \ldots, j_{1,l_{r_{\mathrm{II}}}}\right\}$ with $1 \leqslant l_1 < \ldots < l_{r_{\mathrm{II}}} \leqslant N$. Since $\Delta_{\mathcal{I}_{k+1}}^{\mathcal{J}_{k+1}}(A_{\mathrm{II}}) = 0$ if $|\mathcal{J}_{k+1} \cap [N+1]| = 2$ $(1 \leqslant k \leqslant N)$, then there are $r_{\mathrm{II}} + 1$ and only $r_{\mathrm{II}} + 1$ subsets $\{\mathcal{J}_{k_n+1}\}_{n=1}^{r_{\mathrm{II}}+1}$ of $[3N]$ such that $|\mathcal{J}_{k_n+1} \cap [N+1]| = 1$ for $1 \leqslant n \leqslant r_{\mathrm{II}} + 1$. Consequently, $d_{k_n} = 0$ for $1 \leqslant n \leqslant r_{\mathrm{II}} + 1$, which suggests that $s_{\mathrm{II}} = r_{\mathrm{II}} + 1$.

(c) If $r_{\mathrm{II}} = N$, then $\mathcal{J}_1 \cap [N+1] = \varnothing$. According to the determinantal properties, there must exist some $k^* \in [N]$ such that $|\mathcal{J}_{k^*} \cap [N+1]| = 2$, which causes $\Delta_{\mathcal{I}_{k^*+1}}^{\mathcal{J}_{k^*+1}}(A_{\mathrm{II}}) = 0$. In this case, the disjoint partition $\{\mathcal{J}_{k+1}\}_{k=0}^{N}$ of $[3N]$ does not correspond to any phase combination that actually appears in the determinantal expansion of g (or h). □

For ease of analyzing the dominant behavior of the determinants f, g and h as $t \to \mp\infty$, we follow [23] and define the *asymptotically dominant phase combination* as below.

Definition 4.4. *The phase combinations* $\vartheta_{\mathrm{I}}^{\mathrm{ad}} = \sum_{k=1}^{N}\left(a_k^{\mathrm{ad}}\theta_k + b_k^{\mathrm{ad}}\theta_k^*\right)$ *and* $\vartheta_{\mathrm{II}}^{\mathrm{ad}} = \sum_{k=1}^{N}\left(c_k^{\mathrm{ad}}\theta_k + d_k^{\mathrm{ad}}\theta_k^*\right)$ *are, respectively, said to be asymptotically dominant for the determinants* f *and* g *(or* h*) if* $\mathrm{e}^{\vartheta_{\mathrm{I}}-\vartheta_{\mathrm{I}}^{\mathrm{ad}}} \sim 0$ *or* $\mathrm{e}^{\vartheta_{\mathrm{I}}-\vartheta_{\mathrm{I}}^{\mathrm{ad}}} \sim O(1)$ *and* $\mathrm{e}^{\vartheta_{\mathrm{II}}-\vartheta_{\mathrm{II}}^{\mathrm{ad}}} \sim 0$ *or* $\mathrm{e}^{\vartheta_{\mathrm{II}}-\vartheta_{\mathrm{II}}^{\mathrm{ad}}} \sim O(1)$ *as* $t \to -\infty$ *or* $t \to +\infty$*, where* ϑ_{I} *and* ϑ_{II} *denote any phase combinations appearing in the determinantal expansions of* f *and* g *(or* h*), respectively.*

Theorem 4.5. *Let us define two sign matrices* $\Sigma^- = \left(\sigma_{kj}^-\right)_{N\times N}$ *and* $\Sigma^+ = \left(\sigma_{kj}^+\right)_{N\times N}$ *as follows:*

$$\sigma_{kj}^- = \begin{cases} -1, & k \in \mathcal{S}_j^{(\mathrm{I})} \cup \mathcal{S}_j^{(\mathrm{II})}, \\ 1, & k \in \mathcal{S}_j^{(\mathrm{III})} \cup \mathcal{S}_j^{(\mathrm{IV})}, \\ 0, & k = j, \end{cases} \qquad \sigma_{kj}^+ = \begin{cases} 1, & k \in \mathcal{S}_j^{(\mathrm{I})} \cup \mathcal{S}_j^{(\mathrm{II})}, \\ -1, & k \in \mathcal{S}_j^{(\mathrm{III})} \cup \mathcal{S}_j^{(\mathrm{IV})}, \\ 0, & k = j, \end{cases} \tag{46}$$

where $\mathcal{S}_j^{(\mathrm{I})} = \{m | \mu_m > 0, m = 1, \ldots, j-1\}$, $\mathcal{S}_j^{(\mathrm{II})} = \{m | \mu_m < 0, m = j+1, \ldots, N\}$, $\mathcal{S}_j^{(\mathrm{III})} = \{m | \mu_m < 0, m = 1, \ldots, j-1\}$, $\mathcal{S}_j^{(\mathrm{IV})} = \{m | \mu_m > 0, m = j+1, \ldots, N\}$. *Assume that* $(\beta_k, \gamma_k) \neq (0, 0)$, $\lambda_k \neq 0$, $\lambda_k \neq \lambda_k^*$ $(1 \leqslant k \leqslant N)$ *and* $\{\nu_k\}_{k=1}^{N}$ *are well ordered as* $\nu_1 < \nu_2 < \cdots < \nu_N$. *If* $\xi_j \sim 0$ *for some* $j \in [N]$, *we have the following:*

(i) $\vartheta_{\mathrm{I}}^{\mathrm{ad}}$ *is an asymptotically dominant phase combination of the determinant* f *as* $t \to \mp\infty$ *if and only if* a_k^{ad} *and* b_k^{ad} $(1 \leqslant k \leqslant N)$ *satisfy either of the following two conditions: (a)* $a_j^{\mathrm{ad}} = 1$, $b_j^{\mathrm{ad}} = 2$, $a_k^{\mathrm{ad}} = \sigma_{kj}^{\mp}$ *and* $b_k^{\mathrm{ad}} = 1 + \sigma_{kj}^{\mp}$ *for* $k \neq j$*; (b)* $a_j^{\mathrm{ad}} = -1$, $b_j^{\mathrm{ad}} = 0$, $a_k^{\mathrm{ad}} = \sigma_{kj}^{\mp}$ *and* $b_k^{\mathrm{ad}} = 1 + \sigma_{kj}^{\mp}$ *for* $k \neq j$.

(ii) $\vartheta_{\mathrm{II}}^{\mathrm{ad}}$ *is an asymptotically dominant phase combination of the determinant g (or h) as* $t \to \mp\infty$ *if and only if* c_k^{ad} *and* d_k^{ad} $(1 \leqslant k \leqslant N)$ *satisfy the following conditions:* $c_j^{\mathrm{ad}} = 1$, $d_j^{\mathrm{ad}} = 0$, $c_k^{\mathrm{ad}} = \sigma_{kj}^{\mp}$ *and* $d_k^{\mathrm{ad}} = 1 + \sigma_{kj}^{\mp}$ *for* $k \neq j$.

Proof. Assume that $\vartheta_{\mathrm{I}} = \sum_{k=1}^{N}(a_k\theta_k + b_k\theta_k^*)$ and $\vartheta_{\mathrm{II}} = \sum_{k=1}^{N}(c_k\theta_k + d_k\theta_k^*)$ represent any phase combinations satisfying the properties (i), (ii) and (iii) in lemma 4.3.

(i) *Sufficiency:* if $\vartheta_{\mathrm{I}}^{\mathrm{ad}} = \theta_j + 2\theta_j^* + \sum_{k\neq j}\left[\theta_k^* + \sigma_{kj}^{\mp}(\theta_k + \theta_k^*)\right]$ or $\vartheta_{\mathrm{I}}^{\mathrm{ad}} = -\theta_j + \sum_{k\neq j}\left[\theta_k^* + \sigma_{kj}^{\mp}(\theta_k + \theta_k^*)\right]$, the real part of the difference between ϑ_{I} and $\vartheta_{\mathrm{I}}^{\mathrm{ad}}$ is given by

$$\mathrm{Re}\left(\vartheta_{\mathrm{I}} - \vartheta_{\mathrm{I}}^{\mathrm{ad}}\right) = \sum_{k=1}^{N}\gamma_k\xi_k, \tag{47}$$

where $\gamma_j = a_j + b_j - 3$ or $\gamma_j = a_j + b_j + 1$, $\gamma_k = a_k + b_k - 2\sigma_{kj}^{\mp} - 1$ for $k \neq j$. Because $a_k \in \{-1, 1\}$ and $b_k \in \{0, 2\}$, we know that as $t \to -\infty$, $\gamma_k \in \{0, 2, 4\}$ for $k \in \mathcal{S}_j^{(\mathrm{I})} \cup \mathcal{S}_j^{(\mathrm{II})}$ and $\gamma_k \in \{-4, -2, 0\}$ for $k \in \mathcal{S}_j^{(\mathrm{III})} \cup \mathcal{S}_j^{(\mathrm{IV})}$; as $t \to +\infty$, $\gamma_k \in \{-4, -2, 0\}$ for $k \in \mathcal{S}_j^{(\mathrm{I})} \cup \mathcal{S}_j^{(\mathrm{II})}$ and $\gamma_k \in \{0, 2, 4\}$ for $k \in \mathcal{S}_j^{(\mathrm{III})} \cup \mathcal{S}_j^{(\mathrm{IV})}$. As suggested by lemma 4.2, $\xi_k \sim \mp\infty$ for $k \in \mathcal{S}_j^{(\mathrm{I})} \cup \mathcal{S}_j^{(\mathrm{II})}$ and $\xi_k \sim \pm\infty$ for $k \in \mathcal{S}_j^{(\mathrm{III})} \cup \mathcal{S}_j^{(\mathrm{IV})}$ as $t \to \mp\infty$. Therefore, the asymptotic behavior of $\mathrm{e}^{\vartheta_{\mathrm{I}} - \vartheta_{\mathrm{I}}^{\mathrm{ad}}}$ is dominated by $\mathrm{e}^{\sum_{k\neq j}\gamma_k\xi_k} \sim 0$ if there exists some $k^* \in [N] \setminus \{j\}$ such that $\gamma_{k^*} \neq 0$, or dominated by $\mathrm{e}^{\gamma_j\xi_j} \sim O(1)$ if $\gamma_k = 0$ for any $k \in [N] \setminus \{j\}$.

Necessity: let $\vartheta_{\mathrm{I}}' = a_j'\theta_j + b_j'\theta_j^* + \sum_{k\neq j}\left[\theta_k^* + \sigma_{kj}^{\mp}(\theta_k + \theta_k^*)\right]$ with $a_j' = 1$, $b_j' = 2$, or $a_j' = -1$, $b_j' = 0$. Subtraction of ϑ_{I}' from $\vartheta_{\mathrm{I}}^{\mathrm{ad}}$ gives

$$\mathrm{Re}\left(\vartheta_{\mathrm{I}}' - \vartheta_{\mathrm{I}}^{\mathrm{ad}}\right) = \sum_{k=1}^{N}\gamma_k'\xi_k, \tag{48}$$

where $\gamma_j' = 3 - a_j^{\mathrm{ad}} - b_j^{\mathrm{ad}}$ or $\gamma_j' = 1 - a_j^{\mathrm{ad}} - b_j^{\mathrm{ad}}$, $\gamma_k' = 2\,\sigma_{kj}^{\mp} - a_k^{\mathrm{ad}} - b_k^{\mathrm{ad}} + 1$ $(k \neq j)$ satisfy that as $t \to -\infty$, $\gamma_k' \in \{0, -2, -4\}$ for $k \in \mathcal{S}_j^{(\mathrm{I})} \cup \mathcal{S}_j^{(\mathrm{II})}$ and $\gamma_k' \in \{0, 2, 4\}$ for $k \in \mathcal{S}_j^{(\mathrm{III})} \cup \mathcal{S}_j^{(\mathrm{IV})}$; as $t \to +\infty$, $\gamma_k' \in \{0, 2, 4\}$ for $k \in \mathcal{S}_j^{(\mathrm{I})} \cup \mathcal{S}_j^{(\mathrm{II})}$ and $\gamma_k' \in \{0, -2, -4\}$ for $k \in \mathcal{S}_j^{(\mathrm{III})} \cup \mathcal{S}_j^{(\mathrm{IV})}$. Accordingly, $\vartheta_{\mathrm{I}}^{\mathrm{ad}}$ is an asymptotically dominant phase combination only if $\gamma_k' = 0$ for $k \neq j$, namely, $a_k^{\mathrm{ad}} + b_k^{\mathrm{ad}} = 2\sigma_{kj}^{\mp} + 1$ $(k \neq j)$. Because of the properties (i) and (ii) in lemma 4.3, it can be determined that $a_k^{\mathrm{ad}} = \sigma_{kj}^{\mp}$ and $b_k^{\mathrm{ad}} = \sigma_{kj}^{\mp} + 1$ for $k \neq j$; $a_j^{\mathrm{ad}} = 1$, $b_j^{\mathrm{ad}} = 2$, or $a_j^{\mathrm{ad}} = -1$, $b_j^{\mathrm{ad}} = 0$.

(ii) The proof of part (ii) follows a similar way as that in part (i) of this theorem. Note that c_k^{ad} and d_k^{ad} $(1 \leqslant k \leqslant N)$ obey properties (i) and (iii) in lemma 4.3. That is why there is only one asymptotically dominant phase combination in the determinant g (or h). □

Theorem 4.5 is the main result in this section because it gives an affirmative answer to the question that how to determine the asymptotically dominant terms in the determinants f, g and h as $t \to \mp\infty$. Based on this theorem, the procedure for deriving the asymptotic expressions for the generic bright N-soliton solutions as $t \to \mp\infty$ can be described as follows:

(i) For any given set of spectral parameters $\{\lambda_k\}_{k=1}^{N}$, obtain the sets $\mathcal{S}_j^{(\mathrm{I})}$, $\mathcal{S}_j^{(\mathrm{II})}$, $\mathcal{S}_j^{(\mathrm{III})}$ and $\mathcal{S}_j^{(\mathrm{IV})}$ $(1 \leqslant j \leqslant N)$, and two sign matrices Σ^- and Σ^+.

(ii) As $t \to \mp\infty$, find the coefficients $C_{1j}^{\mp}$ and $C_{2j}^{\mp}$ $(1 \leqslant j \leqslant N)$ that correspond to two asymptotically dominant phase combinations $\vartheta_{\mathrm{I}}^{\mathrm{ad}} = \theta_j + 2\,\theta_j^* + \sum_{k\neq j}\left[\theta_k^* + \sigma_{kj}^{\mp}(\theta_k + \theta_k^*)\right]$ and $\vartheta_{\mathrm{I}}^{\mathrm{ad}} = -\theta_j + \sum_{k\neq j}\left[\theta_k^* + \sigma_{kj}^{\mp}(\theta_k + \theta_k^*)\right]$ in the determinantal expansion of f.

(iii) As $t \to \mp\infty$, find the coefficients $C_{3j}^{\mp}$ and $C_{4j}^{\mp}$ that respectively correspond to the asymptotically dominant phase combination $\vartheta_{\rm II}^{\rm ad} = \theta_j + \sum_{k\neq j}\left[\theta_k^* + \sigma_{kj}^{\mp}(\theta_k + \theta_k^*)\right]$ in the determinantal expansions of g and h.

(iv) The jth asymptotic solitons as $t \to \mp\infty$ can be obtained as follows:

$$u_j^{\mp} = -\frac{C_{3j}^{\mp}\,\mathrm{e}^{\theta_j-\theta_j^*}}{2\sqrt{C_{1j}^{\mp}C_{2j}^{\mp}}}\,\mathrm{sech}\left(\theta_j + \theta_j^* + \ln\sqrt{\frac{C_{1j}^{\mp}}{C_{2j}^{\mp}}}\right), \tag{49}$$

$$v_j^{\mp} = (-1)^{N-1}\frac{C_{4j}^{\mp}\,\mathrm{e}^{\theta_j-\theta_j^*}}{2\sqrt{C_{1j}^{\mp}C_{2j}^{\mp}}}\,\mathrm{sech}\left(\theta_j + \theta_j^* + \ln\sqrt{\frac{C_{1j}^{\mp}}{C_{2j}^{\mp}}}\right). \tag{50}$$

Example: asymptotic expressions of the bright two-soliton solutions. According to the signs of μ_1 and μ_2, the asymptotic analysis falls into four cases: (a) $\mu_1 > 0$, $\mu_2 > 0$; (b) $\mu_1 > 0$, $\mu_2 < 0$; (c) $\mu_1 < 0$, $\mu_2 > 0$; (d) $\mu_1 < 0$, $\mu_2 < 0$. Here, we just take into account the first case since the other three cases can be analyzed in a similar manner. In this case, we first obtain the following results:

$$S_1^{(\rm I)} = S_1^{(\rm II)} = S_1^{(\rm III)} = \varnothing, \qquad S_1^{(\rm IV)} = \{2\},$$
$$S_2^{(\rm I)} = \{1\}, \qquad S_2^{(\rm II)} = S_2^{(\rm III)} = S_2^{(\rm IV)} = \varnothing,$$
$$\Sigma^- = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \Sigma^+ = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Then, the expressions of two asymptotic solitons as $t \to \mp\infty$ are presented as follows:

(1) As $t \to -\infty$:

$$u_1^- = \frac{\mathrm{i}\beta_1^*(\lambda_1-\lambda_1^*)(\lambda_2-\lambda_1^*)(\lambda_1-\lambda_2)\,\mathrm{e}^{\theta_1-\theta_1^*}\,\mathrm{sech}\left[\theta_1+\theta_1^*+\ln\left(\frac{|\lambda_1-\lambda_2|}{|\lambda_1-\lambda_2^*|\sqrt{|\beta_1|^2+|\gamma_1|^2}}\right)\right]}{|\lambda_1-\lambda_2||\lambda_1^*-\lambda_2|\sqrt{|\beta_1|^2+|\gamma_1|^2}}, \tag{51}$$

$$v_1^- = \frac{\mathrm{i}\gamma_1^*(\lambda_1-\lambda_1^*)(\lambda_2-\lambda_1^*)(\lambda_1-\lambda_2)\,\mathrm{e}^{\theta_1-\theta_1^*}\,\mathrm{sech}\left[\theta_1+\theta_1^*+\ln\left(\frac{|\lambda_1-\lambda_2|}{|\lambda_1-\lambda_2^*|\sqrt{|\beta_1|^2+|\gamma_1|^2}}\right)\right]}{|\lambda_1-\lambda_2||\lambda_1^*-\lambda_2|\sqrt{|\beta_1|^2+|\gamma_1|^2}}, \tag{52}$$

$$u_2^- = \frac{\mathrm{i}\kappa_1(\lambda_2-\lambda_2^*)(\lambda_2-\lambda_1^*)\,\mathrm{e}^{\theta_2-\theta_2^*}\,\mathrm{sech}\left[\theta_2+\theta_2^*+\ln\left(\frac{|\lambda_1-\lambda_2^*|\sqrt{|\beta_1|^2+|\gamma_1|^2}}{\kappa_0}\right)\right]}{\kappa_0|\lambda_1-\lambda_2^*|\sqrt{|\beta_1|^2+|\gamma_1|^2}}, \tag{53}$$

$$v_2^- = \frac{\mathrm{i}\kappa_2(\lambda_2-\lambda_2^*)(\lambda_2-\lambda_1^*)\,\mathrm{e}^{\theta_2-\theta_2^*}\,\mathrm{sech}\left[\theta_2+\theta_2^*+\ln\left(\frac{|\lambda_1-\lambda_2^*|\sqrt{|\beta_1|^2+|\gamma_1|^2}}{\kappa_0}\right)\right]}{\kappa_0|\lambda_1-\lambda_2^*|\sqrt{|\beta_1|^2+|\gamma_1|^2}}, \tag{54}$$

with

$$\kappa_0 = \sqrt{(|\beta_1|^2+|\gamma_1|^2)(|\beta_2|^2+|\gamma_2|^2)|\lambda_1-\lambda_2^*|^2+|\beta_1\beta_2^*+\gamma_1\gamma_2^*|^2(\lambda_1-\lambda_1^*)(\lambda_2-\lambda_2^*)},$$
$$\kappa_1 = \gamma_1\lambda_1(\beta_2^*\gamma_1^*-\beta_1^*\gamma_2^*)+\beta_1^*\lambda_1^*(\beta_1\beta_2^*+\gamma_1\gamma_2^*)-\beta_2^*\lambda_2^*(|\beta_1|^2+|\gamma_1|^2),$$
$$\kappa_2 = \gamma_1^*\lambda_1^*(\beta_1\beta_2^*+\gamma_1\gamma_2^*)-\beta_1\lambda_1(\beta_2^*\gamma_1^*-\beta_1^*\gamma_2^*)-\gamma_2^*\lambda_2^*(|\beta_1|^2+|\gamma_1|^2).$$

(2) As $t \to +\infty$:

$$u_1^+ = \frac{\mathrm{i}\,\kappa_3(\lambda_1-\lambda_1^*)(\lambda_2^*-\lambda_1)\,\mathrm{e}^{\theta_1-\theta_1^*}\,\mathrm{sech}\big[\theta_1+\theta_1^*+\ln\big(\frac{|\lambda_2-\lambda_1^*|\sqrt{|\beta_2|^2+|\gamma_2|^2}}{\kappa_0}\big)\big]}{\kappa_0|\lambda_2-\lambda_1^*|\sqrt{|\beta_2|^2+|\gamma_2|^2}}, \tag{55}$$

$$v_1^+ = \frac{\mathrm{i}\kappa_4(\lambda_1-\lambda_1^*)(\lambda_2^*-\lambda_1)\,\mathrm{e}^{\theta_1-\theta_1^*}\,\mathrm{sech}\big[\theta_1+\theta_1^*+\ln\big(\frac{|\lambda_2-\lambda_1^*|\sqrt{|\beta_2|^2+|\gamma_2|^2}}{\kappa_0}\big)\big]}{\kappa_0|\lambda_2-\lambda_1^*|\sqrt{|\beta_2|^2+|\gamma_2|^2}}, \tag{56}$$

$$u_2^+ = \frac{\mathrm{i}\beta_2^*(\lambda_2^*-\lambda_1)(\lambda_2-\lambda_2^*)(\lambda_1-\lambda_2)\,\mathrm{e}^{\theta_2-\theta_2^*}\,\mathrm{sech}\big[\theta_2+\theta_2^*+\ln\big(\frac{|\lambda_1-\lambda_2|}{|\lambda_1-\lambda_2^*|\sqrt{|\beta_2|^2+|\gamma_2|^2}}\big)\big]}{|\lambda_1-\lambda_2||\lambda_1-\lambda_2^*|\sqrt{|\beta_2|^2+|\gamma_2|^2}}, \tag{57}$$

$$v_2^+ = \frac{\mathrm{i}\gamma_2^*(\lambda_2^*-\lambda_1)(\lambda_2-\lambda_2^*)\,(\lambda_1-\lambda_2)\,\mathrm{e}^{\theta_2-\theta_2^*}\,\mathrm{sech}\big[\theta_2+\theta_2^*+\ln\big(\frac{|\lambda_1-\lambda_2|}{|\lambda_1-\lambda_2^*|\sqrt{|\beta_2|^2+|\gamma_2|^2}}\big)\big]}{|\lambda_1-\lambda_2||\lambda_1-\lambda_2^*|\sqrt{|\beta_2|^2+|\gamma_2|^2}}, \tag{58}$$

with

$$\kappa_3 = \gamma_2\lambda_2(\beta_2^*\gamma_1^*-\beta_1^*\gamma_2^*)+\beta_1^*\lambda_1^*(|\beta_2|^2+|\gamma_2|^2)-\beta_2^*\lambda_2^*(\beta_2\beta_1^*+\gamma_2\gamma_1^*),$$
$$\kappa_4 = \gamma_1^*\lambda_1^*(|\beta_2|^2+|\gamma_2|^2)-\beta_2\lambda_2(\beta_2^*\gamma_1^*-\beta_1^*\gamma_2^*)-\gamma_2^*\lambda_2^*(\beta_2\beta_1^*+\gamma_2\gamma_1^*).$$

5. Concluding remarks

Currently, intensive attention both in the theory and experiment has been paid to the coupled NLS equations which have a variety of applications in optics such as the wavelength-division-multiplexed optical soliton transmission [30], multi-channel bit parallel-wavelength optical fiber networks [31] and all-optical switching, logic and computation [32, 33]. In this paper, we have constructed the triple Wronskian solutions to the coupled NLS equations (2*a*) and (2*b*) by iterating the DT-based algorithm. Some new determinantal properties of the triple Wronskian have been derived in lemmas 3.2 and 3.3 to complete the proof. With a set of N linearly independent solutions of the zero-potential Lax pair, we have found that the triple Wronskian solutions give rise to the bright N-soliton solutions characterized by $3N$ complex parameters. For any given set of spectral parameters $\{\lambda_k\}_{k=1}^N$, we have derived an algorithmic method in theorem 4.5 to identify the asymptotically dominant terms in the expansion of the triple Wronskian. Furthermore, we have presented an algebraic procedure of obtaining the asymptotic expressions for the bright N-soliton solutions with arbitrary N as $t \to \mp\infty$. Finally, the expressions of asymptotic solitons of the bright two-soliton solutions have been given as an illustrative example.

In future, we plan to continue our work along the following two directions.

(1) The coupled NLS solitons are known to admit the shape-change collision which has attracted much interest due to its applications in the construction of the logic gates [32] and the design of the turing-equivalent all-optical computing machines [33]. Accordingly, it becomes practically important to clarify the asymptotic behavior of the coupled NLS multi-soliton solutions. The procedure presented in section 4 enables us to obtain the quantities such as the amplitudes, energies, velocities and widths to characterize asymptotic solitons of the generic two-component bright N-soliton solutions.

(2) We conjecture that the m-coupled NLS equations of the Manakov type admit the bright N-soliton solutions in terms of the $(m + 1)$-component Wronskian. Nonetheless, it might not be easy to finish the proof because some new identity relations of the $(m + 1)$-component Wronskian require to be derived. If this conjecture is correct, we can also obtain a procedure of deriving the asymptotic expressions for the generic bright N-soliton solutions to the m-coupled NLS equations based on the $(m + 1)$-component Wronskian structure.

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Appendix

The triple Wronskian notations of f_x, g_x, h_x, f_t, g_t, h_t, f_{xx}, g_{xx}, h_{xx}:

$$f_x = |\widehat{N-2}, N; \widehat{N-1}; \widehat{N-1}| + |\widehat{N-1}; \widehat{N-2}, N; \widehat{N-1}| + |\widehat{N-1}; \widehat{N-1}; \widehat{N-2}, N|, \tag{A.1}$$

$$g_x = 2|\widehat{N-1}, N+1; \widehat{N-2}; \widehat{N-1}| + |\widehat{N}; \widehat{N-3}, N-1; \widehat{N-1}| + |\widehat{N}; \widehat{N-2}; \widehat{N-2}, N|, \tag{A.2}$$

$$h_x = 2|\widehat{N-1}, N+1; \widehat{N-1}; \widehat{N-2}| + |\widehat{N}; \widehat{N-2}, N; \widehat{N-2}| + |\widehat{N}; \widehat{N-1}; \widehat{N-3}, N-1|, \tag{A.3}$$

$$\begin{aligned} f_t = 2\,\mathrm{i}(&|\widehat{N-2}, N+1; \widehat{N-1}; \widehat{N-1}| - |\widehat{N-3}, N-1, N; \widehat{N-1}; \widehat{N-1}| \\ &+ |\widehat{N-1}; \widehat{N-3}, N-1, N; \widehat{N-1}| - |\widehat{N-1}; \widehat{N-2}, N+1; \widehat{N-1}| \\ &+ |\widehat{N-1}; \widehat{N-1}; \widehat{N-3}, N-1, N| - |\widehat{N-1}; \widehat{N-1}; \widehat{N-2}, N+1|), \end{aligned} \tag{A.4}$$

$$\begin{aligned} g_t = 4\,\mathrm{i}(&|\widehat{N-1}, N+2; \widehat{N-2}; \widehat{N-1}| - |\widehat{N-2}, N, N+1; \widehat{N-2}; \widehat{N-1}| \\ &+ |\widehat{N}; \widehat{N-4}, N-2, N-1; \widehat{N-1}| - |\widehat{N}; \widehat{N-3}, N; \widehat{N-1}| \\ &+ |\widehat{N}; \widehat{N-2}; \widehat{N-3}, N-1, N| - |\widehat{N}; \widehat{N-2}; \widehat{N-2}, N+1|), \end{aligned} \tag{A.5}$$

$$\begin{aligned} h_t = 4\,\mathrm{i}(&|\widehat{N-1}, N+2; \widehat{N-1}; \widehat{N-2}| - |\widehat{N-2}, N, N+1; \widehat{N-1}; \widehat{N-2}| \\ &+ |\widehat{N}; \widehat{N-1}; \widehat{N-4}, N-2, N-1| - |\widehat{N}; \widehat{N-1}; \widehat{N-3}, N| \\ &+ |\widehat{N}; \widehat{N-3}, N-1, N; \widehat{N-2}| - |\widehat{N}; \widehat{N-2}, N+1; \widehat{N-2}|), \end{aligned} \tag{A.6}$$

$$\begin{aligned} f_{xx} = {} & |\widehat{N-3}, N-1, N; \widehat{N-1}; \widehat{N-1}| + |\widehat{N-2}, N+1; \widehat{N-1}; \widehat{N-1}| \\ & + 2|\widehat{N-2}, N; \widehat{N-2}, N; \widehat{N-1}| + |\widehat{N-1}; \widehat{N-3}, N-1, N; \widehat{N-1}| \\ & + |\widehat{N-1}; \widehat{N-2}, N+1; \widehat{N-1}| + 2|\widehat{N-1}; \widehat{N-2}, N; \widehat{N-2}, N| \\ & + 2|\widehat{N-2}, N; \widehat{N-1}; \widehat{N-2}, N| + |\widehat{N-1}; \widehat{N-1}; \widehat{N-3}, N-1, N| \\ & + |\widehat{N-1}; \widehat{N-1}; \widehat{N-2}, N+1|, \end{aligned} \tag{A.7}$$

$$\begin{aligned} g_{xx} = {} & 2\,(|\widehat{N-2}, N, N+1; \widehat{N-2}; \widehat{N-1}| + |\widehat{N-1}, N+2; \widehat{N-2}; \widehat{N-1}| \\ & + 2|\widehat{N-1}, N+1; \widehat{N-3}, N-1; \widehat{N-1}| + |\widehat{N}; \widehat{N-2}; \widehat{N-2}, N+1| \\ & + 2|\widehat{N-1}, N+1; \widehat{N-2}; \widehat{N-2}, N| + |\widehat{N}; \widehat{N-3}, N; \widehat{N-1}| \\ & + 2|\widehat{N}; \widehat{N-3}, N-1; \widehat{N-2}, N| + |\widehat{N}; \widehat{N-2}; \widehat{N-3}, N-1, N| \\ & + |\widehat{N}; \widehat{N-4}, N-2, N-1; \widehat{N-1}|), \end{aligned} \tag{A.8}$$

$$\begin{aligned} h_{xx} = {} & 2\,(|\widehat{N-2}, N, N+1; \widehat{N-1}; \widehat{N-2}| + |\widehat{N-1}, N+2; \widehat{N-1}; \widehat{N-2}| \\ & + 2|\widehat{N-1}, N+1; \widehat{N-1}; \widehat{N-3}, N-1| + |\widehat{N}; \widehat{N-2}, N+1; \widehat{N-2}| \\ & + 2|\widehat{N-1}, N+1; \widehat{N-2}, N; \widehat{N-2}| + |\widehat{N}; \widehat{N-1}; \widehat{N-3}, N| \\ & + 2|\widehat{N}; \widehat{N-2}, N; \widehat{N-3}, N-1| + |\widehat{N}; \widehat{N-3}, N-1, N; \widehat{N-2}| \\ & + |\widehat{N}; \widehat{N-1}; \widehat{N-4}, N-2, N-1|). \end{aligned} \tag{A.9}$$

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